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The Inhomogeneous Invariance Quantum Group of Particle Algebras with Continuous Parameters

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Abstract We construct the inhomogeneous quantum groups with continuous parameters which leave the boson and fermion algebras of field theory invariant.

Keywords Quantum groups with continuous parameters · Hopf algebra

1 Introduction

The idea of quantum groups came from quantization of inverse scattering method to study the behavior of integrable systems in quantum field theory [1]. Quantum groups are algebraic objects which contain Lie algebras and Lie groups as special cases. There are two main approaches to construct a quantum group. The first method is deformation. In this method a deformation is made on a Lie group. Usually the deformation parameter is shown by q and it takes values between 0 and 1 [2]. As an example the quantum algebra $U_q(su_2)$ is the associative algebra generated by the elements H, E_+, E_- that obey the commutation relations

$$\begin{aligned} [H, E_+] &= E_+, \\ [H, E_-] &= -E_-, \\ [E_+, E_-] &= \frac{q^H - q^{-H}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \end{aligned} \tag{1}$$

Here $E_+^* = E_-$ and $H^* = H$, also the indices are discrete.

The second method gives matrix quantum groups [3]. In this method one can build up a matrix whose elements satisfy Hopf algebra axioms. In this paper we use the term "invariance quantum group" to describe a Hopf algebra such that a particle algebra forms

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a right module of the Hopf algebra. For example the q-deformed Pusz-Woronowicz algebra is a q-deformation of the harmonic oscillator algebra and possesses the commutation relations [4]

$$\begin{aligned} a_i a_j - q a_j a_i &= 0, \quad \text{for } i < j, \\ a_i^* a_j^* - q^{-1} a_j^* a_i^* &= 0, \quad \text{for } i < j, \\ a_i a_j^* - q a_j^* a_i &= 0, \quad \text{for } i \neq j, \\ a_i a_i^* - q^2 a_i^* a_i &= 1 + (q^2 - 1) \sum_{j=1}^{i-1} a_j^* a_j. \end{aligned} \tag{2}$$

This algebra is a right module of the q-deformed matrix quantum group $SU_q(2)$. Here the indices are discrete, if one generalizes to the continuous case the algebra becomes inconsistent, since for example

$$a(p)a(p') = q a(p')a(p), \tag{3}$$

in the limit of $p = p'$, (3) is inconsistent. Thus quantum groups which are built by the deformation method are usually not continuous algebraic objects.

Experimentally observed particles are classified as bosons and fermions. Their behaviour can be described via algebraic relations, they are fermion and boson algebras. The algebras can be written as;

$$\left[c_i, c_j^*\right]_\pm = \delta_{ij}, \tag{4}$$

and

$$\left[c_i, c_j\right]_\pm = 0. \tag{5}$$

The upper sign is for fermions and the lower sign is for bosons. Here i and j are discrete indices and they take values from 1 to d, where d is number of bosons (fermions). Their inhomogeneous quantum groups are *BISp*, Bosonic inhomogeneous symplectic [5] and *FIO*, fermionic inhomogeneous orthogonal [6], quantum groups. These two quantum groups are matrix quantum groups and the elements of the matrix are discrete quantities.

Making an inhomogeneous linear canonical transformation on the particle algebras one can find their inhomogeneous symmetry groups. For the bosons the inhomogeneous symmetry group is the inhomogeneous symplectic group $ISp(2d, \mathbb{R})$ [7]. For the fermions the inhomogeneous generalization becomes a supergroup which can be denoted by $GrIO(2d, \mathbb{R})$ and called the Grassmanian inhomogeneous orthogonal group. Here d is number of particles and $\mathbb{R}$ means real number. One can find a generalization for $\mathbb{C}$ case [8]. Also other symmetry group for boson and fermion algebras can be found using continuous parameters in the transformation [9]. In order to do that the fermion and boson algebras are written in terms of continuous parameter p. Then a question arises, whether there are inhomogeneous linear canonical transformations with continuous parameters which satisfy Hopf algebra axioms, namely which are quantum groups.

2 $FIO(\mathfrak{C}, \mathbb{R})$ and $BISp(\mathfrak{C}, \mathbb{R})$

The concept of bosons and fermions are crucial for the determination of microscopic scale physics. Bosons and fermions are described in terms of creation and annihilation operators and they satisfy the following algebra,

$$\left[c(p), c(p')\right]_{\pm} = 0, \tag{6}$$

$$\left[c(p), c^*(p')\right]_{\pm} = \delta(p - p'), \tag{7}$$

where the upper sign is for the fermion algebra, the lower sign is for the boson algebra. Also $p,\ p' \in \mathbb{R}$ and these are thought as momentum. We consider linear transformation on the creation and annihilation operators of particle algebras. The transformed operators can be written as,

$$c(p)' = \int \phi(p,k) \otimes c(k)dk + \int \psi(p,k) \otimes c^*(k)dk + f(p) \otimes 1, \tag{8}$$

$$c^*(p)' = \int \psi(p,k)^{\dagger} \otimes c(k)dk + \int \phi(p,k)^{\dagger} \otimes c^*(k)dk + f(p)^{\dagger} \otimes 1. \tag{9}$$

The parameters $\psi(p,k)$ and $\phi(p,k)$ are not necessarily commutative homogeneous parameters, $f(p)$ is an noncommutative inhomogeneous parameter. Also the transformation can be symbolically written in matrix form

$$M = \left(\begin{array}{cc|c} \phi & \psi & f \\ \psi^{\dagger} & \phi^{\dagger} & f^{\dagger} \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} A & F \\ \hline 0 & 1 \end{array}\right). \tag{10}$$

Here A is homogeneous part and F is inhomogeneous part of the transformation. We want that the transformation is canonical, which means transformed operators satisfy the same commutation relations as in (6) and (7). That means the algebra remains invariant under the transformation.

To leave the particle algebra unchanged there are commutation relations between the elements of transformation. The homogeneous parameters $\psi(p,k)$, $\phi(p,k)$, $\psi^{\dagger}(p,k)$ and $\phi^{\dagger}(p,k)$ commute with each other. Namely,

$$\left[A_{ij}, A_{kl}\right] = 0. \tag{11}$$

The relation between homogeneous and inhomogeneous parameters are,

$$\left[A_{ij}, F_k\right]_{\pm} = 0, \tag{12}$$

where plus sign for fermionic case and the minus sign for the Bosonic case. Also the relations between the inhomogeneous parameters are

$$\left[f(p), f(p')\right]_{\pm} = -\int \phi(p,k)\psi(p',k)dk \mp \int \phi(p',k)\psi(p,k)dk, \tag{13}$$

$$\left[f(p), f(p')^{\dagger}\right]_{\pm} = \delta(p-p') - \int \phi(p,k)\phi(p',k)^{\dagger}dk \mp \int \psi(p,k)\psi(p',k)^{\dagger}dk, \tag{14}$$

where the upper sign is for fermions and lower sign is for bosons.

In order to know whether the transformation is a quantum group one should look at Hopf algebra axioms. In order to do that the coproduct, counit and antipode must be found.

The coproduct can be defined using matrix multiplication rule. Then the coproduct of the elements of transformation can be found using

$$\Delta(M) = M \dot{\otimes} M.$$

$$\begin{aligned}
\Delta(\phi(p,k)) &= \int \phi(p,\eta)\otimes\phi(\eta,k)d\eta + \int \psi(p,\eta)\otimes\psi^\dagger(\eta,k)d\eta,\\
\Delta(\psi(p,k)) &= \int \phi(p,\eta)\otimes\psi(\eta,k)d\eta + \int \psi(p,\eta)\otimes\phi^\dagger(\eta,k)d\eta,\\
\Delta(f(p)) &= \int \phi(p,\eta)\otimes f(\eta)d\eta + \int \psi(p,\eta)\otimes f^\dagger(\eta)d\eta + f(p)\otimes 1,\\
\Delta(\phi^\dagger(p,k)) &= \int \psi^\dagger(p,\eta)\otimes\psi(\eta,k)d\eta + \int \phi^\dagger(p,\eta)\otimes\phi^\dagger(\eta,k)d\eta,\\
\Delta(\psi^\dagger(p,k)) &= \int \psi^\dagger(p,\eta)\otimes\phi(\eta,k)d\eta + \int \phi^\dagger(p,\eta)\otimes\psi^\dagger(\eta,k)d\eta,\\
\Delta(f^\dagger(p)) &= \int \phi^\dagger(p,\eta)\otimes f^\dagger(\eta)d\eta + \int \psi^\dagger(p,\eta)\otimes f(\eta)d\eta + f^\dagger(p)\otimes 1.
\end{aligned} \tag{15}$$

The coproduct of the relations are satisfied using (15).

The counit is

$$\epsilon(M) = \mathcal{I}, \tag{16}$$

where $\mathcal{I}$ is the unit matrix. Then the antipode can be found taking the inverse of the transformation matrix.

$$M^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}F \\ 0 & 1 \end{pmatrix}, \tag{17}$$

where A^{-1} is defined in the standard way since matrix elements of A are commutative.

The coproduct, the counit and the antipode of the transformation matrix M as given by (15–17) have been constructed. Thus, the elements of M is an element of a quantum group. These quantum groups are the fermionic inhomogeneous orthogonal quantum group with continuous parameters $FIO(\mathfrak{C}, \mathbb{R})$ and the Bosonic inhomogeneous symplectic quantum group with continuous parameters $BISp(\mathfrak{C}, \mathbb{R})$.

3 Conclusion

Although quantum groups are algebraic objects, they play an important role in understanding the quantization of continuous systems. One way to get a quantum group is making a deformation on a usual Lie group. Another way to build a quantum group is writing a matrix such that the Hopf algebra coproduct is given by the matrix product which satisfy Hopf algebra axioms, because quantum groups are Hopf algebras as mathematical language. This type of quantum groups are called matrix quantum groups.

Particle algebras are important since they describe the behavior of microscopic physics. Their symmetry groups are the well known symplectic and orthogonal groups. These groups can be extended to quantum groups. For d creation and annihilation operators the inhomogeneous invariance quantum groups are $BISp(2d)$ and $FIO(2d)$.

In this paper we have investigated whether there is quantum inhomogeneous invariance quantum groups for particle algebras with continuous parameters. We have shown that this quantum group is for bosons $BISp(\mathfrak{C}, \mathbb{R})$ and for fermions $FIO(\mathfrak{C}, \mathbb{R})$.

In field theory a field $\varphi(x)$ can be written using creation and annihilation operators $c(p)$, $c(p)^*$ and vice versa. Since the c and c^* are operators, the field $\varphi(x)$ also an operator. Using transformed creation and annihilation operators one can write a transformed field and find a quantum group for field $\varphi(x)$.

The theory of quantum groups can be helpful in generalization of quantization and one can get more consistent approach to interacting field theories.

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